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ON A GAS SOURCE IN A CONSTANT FORCE FIELD

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The nonbarochronic regular partially invariant submodel of the equations of gas dynamics is studied. The submodel reduces to an implicit ordinary differential equation of the first order for an auxiliary function $X = X(x)$. The physical quantities (velocity, density, and pressure) are expressed in terms of the function X. The properties of the solutions of the equation are investigated and interpreted physically in terms of gas motion. The existence of a shock-wave solution is proved. The properties of the shock adiabat are studied. It is shown that the results obtained are new and differ significantly from the results for the case of no constant force.

Key words: *partially invariant solution, discriminant curve, jet space, irregular singular point, projective change, sonic line, stationary shock wave.*

Introduction. Group analysis of differential equations [1] is an effective method for constructing broad classes of exact solutions of models of continuum mechanics, in particular, gas dynamics. Ovsyannikov [2] studied the exact solution of the equations of gas dynamics that describes the two-dimensional motion of a gas in a force field with a constant acceleration (gravity). This motion is generated by the regular partially invariant submodel defined by 4-dimensional algebra with the addition of the external force to the first equation of momentum. Chupakhin [3] described the corresponding submodel for the case where the force is absent. Stanyukovich [4] considered gas motion in the presence of external potential forces described by a simple wave. The proposed solution does not reduce to a simple wave and is a new one.

The solutions corresponding to different regimes of gas motion for different ratios of kinetic and potential energies are studied. The mathematical model reduces to an implicit differential equation of the first order. The properties of similar equations are described by Arnol'd [5].

1. Description of the Model. The algebra generating the solution has the basis $L_4 = \langle \partial_y, \partial_z, t\partial_y + \partial_v, \partial_t \rangle$. The invariants of this submodel are x, u, w, ρ, p, and S, where u and w are the velocity components; the thermodynamic parameters ρ, p, and S are the density, pressure, and entropy, respectively. The superfluous function is the velocity component v. The solution is represented as

$$u = u(x), \qquad v = v(t, x, y, z), \qquad w = w(x), \qquad (\rho, S, p) \mid x.$$

The equations of the submodel are written as

$$\begin{gathered} uu' + \rho^{-1}p' = g_0, \qquad v_t + uv_x + vv_y + wv_x = 0, \qquad uw' = 0, \\ u\rho' + \rho(u' + v_y) = 0, \qquad uS' = 0, \end{gathered} \tag{1.1}$$

where $g_0 = \text{const}$ and $g_0 \neq 0$; the prime denotes differentiation with respect to x.

Let us consider the case $u \neq 0$. In this case, submodel (1.1) describes isentropic gas motion $S = s_0 = \text{const}$. In addition, the third equation of (1.1) implies that $w = W_0 = \text{const}$, and the fourth equation implies the representation

$$v = h(x)y + V(t, x, z), \tag{1.2}$$

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where

$$h = -(u(\ln\rho)' + u'). \tag{1.3}$$

Substituting representation (1.2) into the second equation of (1.1) and splitting the resulting relations in y, we obtain

$$uh' + h^2 = 0; \tag{1.4}$$

$$V_t + uV_x + W_0V_x + hV = 0. \tag{1.5}$$

The first and third equations in (1.1) and Eqs. (1.3) and (1.4) form an invariant subsystem. The overdetermined system for the noninvariant component which includes the second and fourth equations of (1.1) is split into the invariant equation (1.4) and Eq. (1.5) for the noninvariant part. After integration of the invariant subsystem, Eq. (1.5) is integrated as a linear equation.

Integration of the invariant system can be reduced to solving a first-order ordinary differential equation and several quadratures. We introduce the function $\sigma = \sigma(x)$ ($\sigma \neq$ const) such that $\sigma = 1/h$. Then, Eq. (1.4) becomes $u\sigma' = 1$ and we obtain the representation

$$u = 1/\sigma', \qquad h = 1/\sigma. \tag{1.6}$$

In terms of the function $\sigma(x)$, the continuity equation (1.3) is written as $(\ln\rho)' - \sigma''/\sigma' + \sigma'/\sigma = 0$ and is integrated:

$$\rho = R_0|\sigma'/\sigma|, \qquad R_0 = \text{const}. \tag{1.7}$$

The function $\sigma = \sigma(x)$ is a solution of the first equation of momentum in (1.1), and integrating it, we obtain the invariant Bernoulli integral:

$$u^2/2 + I(\rho) = g_0x + b_0. \tag{1.8}$$

Here

$$I(\rho) = \int dp/\rho$$

is the enthalpy of the gas. For a polytropic gas, $p = S_0\rho^\gamma$ (the motion is isentropic). Substitution of (1.6) and (1.7) into (1.8) yields

$$\frac{1}{2(\sigma')^2} + \frac{c_0^2}{\gamma-1}\Big|\frac{\sigma'}{\sigma}\Big|^{\gamma-1} = g_0x + b_0, \tag{1.9}$$

where $c_0^2 = \gamma S_0R_0^{\gamma-1} = \text{const}$.

The integrals of Eqs. (1.5) are expressed in terms of the function $\sigma = \sigma(x)$ by finite formulas. As a result, we have a solution of the form

$$u = \frac{1}{\sigma'}, \quad v = \frac{y + H(\xi,\eta)}{\sigma}, \quad w = W_0, \quad \rho = R_0\Big|\frac{\sigma'}{\sigma}\Big|, \quad S = S_0, \quad p = S_0\rho^\gamma, \tag{1.10}$$

where H is an arbitrary function of the arguments $\xi = t - \sigma(x)$ and $\eta = z - W_0t$ and W_0, R_0, and S_0 are arbitrary constants. The function $\sigma = \sigma(x)$ satisfies Eq. (1.9).

2. Key Equation. Equation (1.9) can be written as

$$(\sigma')^2\Big|\frac{\sigma'}{\sigma}\Big|^{\gamma-1} - \frac{g_0(\gamma-1)}{c_0^2}\Big(x + \frac{b_0}{g_0}\Big)(\sigma')^2 + \frac{\gamma-1}{2c_0^2} = 0. \tag{2.1}$$

Theorem 1. *The dimension of the bundle of integral curves for the key equation* (2.1) *does not exceed four for an arbitrary rational exponent* $\gamma > 1$.

Proof. The key equation (2.1) for any rational exponents γ is an algebraic equation for the derivative or is reduced to this by a change of variables.

The number of positive real roots can be estimated using the Descartes' rule, according to which the number of positive real roots of a polynomial does not exceed the number of sign changes in the sequence of its coefficients [6].

Taking into account that the rational number $\gamma > 1$, we consider all possible cases:

$$\gamma = 2m, \quad \gamma = 2m+1, \quad \gamma = \frac{2m}{2n+1}, \quad \gamma = \frac{2m+1}{2n}, \quad \gamma = \frac{2m+1}{2n+1} \tag{2.2}$$

(m and n are natural numbers).

1. In (2.2), let $\gamma = 2m$. Then, the key equation (2.1) is written as

$$(\sigma')^2\left|\frac{\sigma'}{\sigma}\right|^{2m-1} - \frac{g_0(2m-1)}{c_0^2}\left(x + \frac{b_0}{g_0}\right)(\sigma')^2 + \frac{2m-1}{2c_0^2} = 0. \tag{2.3}$$

The region of existence of the solution is divided into two parts: a region in which $\sigma\sigma' \geq 0$ and a region in which $\sigma\sigma' < 0$. In each of these regions, the modulus is uncovered by two methods. We write the sequence of signs at the corresponding powers of the derivative depending on the signs of σ and σ':

— If $\sigma\sigma' \geq 0$, then $(+,-,+)$;

— If $\sigma\sigma' < 0$, then $(-,-,+)$.

In each of the above regions, the key equation has no more than two positive roots. To count the number of negative roots, it is necessary to make the change $\sigma' \to -\sigma'$. In this case, however, the number of negative roots is also not larger than two; consequently, the key equation has no more than four real roots in the region of existence of the solution, and for it there exist no more than four integral curves passing through the same point. Theorem 1 is proved.

2. If $\gamma = 2m+1$, the sequence of signs is retained.

3. If $\gamma = 2m/(2n+1)$, the change $(\sigma')^{1/(2n+1)} \to q$ reduces the key equation to Eq. (2.3), for which the required statement was already proved.

4. If $\gamma = (2m+1)/(2n)$, the change $(\sigma')^{1/(2n)} \to q$ in the key equation again leads to Eq. (2.3). It should be noted that this changes involves extracting a root of even order but this does not affect the number of roots since the property of having fixed sign for the function σ was already taken into account.

5. If $\gamma = (2m+1)/(2n+1)$, the change $(\sigma')^{1/(2n+1)} \to q$ reduces the key equation to Eq. (2.3), for which the required statement was proved.

Thus, for all possible rational exponents γ, the dimension of the bundle of integral curves of the key equation (2.1) does not exceed four.

For definiteness, we consider the case $\gamma = 3$. Making the change

$$x + b_0/g_0 \to x, \qquad \sigma \to |1/c_0|X$$

in Eq. (2.1) and denoting $p = dX/dx$, we obtain

$$F(x, X, p) \equiv p^4 - 2\alpha_0^2 x X^2 p^2 + X^2 = 0, \tag{2.4}$$

where $\alpha_0^2 = g_0/c_0^2$. Since the quantity $g_0 x$ is positive, g_0 and x should have the same sign. If $g_0 < 0$, the change $x \to -x$ reduces the given case to the case $x > 0$. Moreover, the change $X' \to -X'$ does not alter the form of the key equation. Thus, without loss of generality, we can assume that $g_0 > 0$. Equation (2.4) will be called the key equation. It belongs to the class of implicit differential equations, is not integrated in quadratures, and has a number of special properties [5].

3. Properties of the Solution of the Key Equation. Equation (2.4) defines a surface in the jet space $\mathbb{R}^3(x, X, p)$, which will be called the equation surface (Fig. 1). The equation surface consists of four isolated components which are symmetric about the Ox axis in the different octants of the space $\mathbb{R}^3(x, X, p)$. Equation (2.4) is determined for $x > 0$. Each of the components is a two-ply sheet above the plane $p = 0$.

Lemma 1. *The geometry of the key-equation surface in the space of* 1*-jets does not change with variation in the problem parameter* α_0.

Proof. We write Eq. (2.4) as

$$p = \varepsilon_1|\alpha_0|\sqrt{xX^2 + \varepsilon_0 X\sqrt{x^2X^2 - 1/\alpha_0^4}}.$$

The parameter $|\alpha_0|$ outside the radical sign does not influence the geometry of the surface; it only changes the scale in the vertical direction. The parameter α_0 under the radical sign influences only the position of the discriminant curve by increasing or decreasing the distance from it to the Ox and OX axes; therefore, the equation

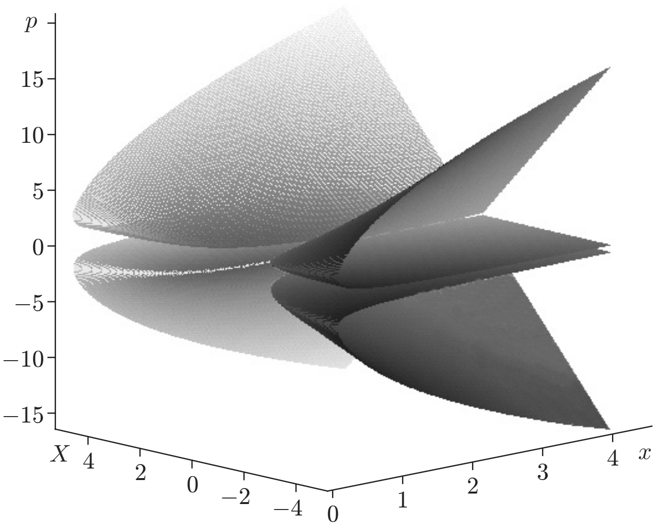

Fig. 1. Surface determined by the key equation (2.4) in the jet space.

surface changes by shifting parallel to the plane $x = 0$ in the direction $x > 0$ in the case of a decrease in the parameter and in the direction $x = 0$ in case of its increase. The composition of a shift and stretching does not change the geometry of the equation surface.

Lemma 2. *All solutions of Eq.* (2.4), *except for* $X \equiv 0$, *are strictly monotonic functions of the variable* x.

Proof. Indeed, if $p = 0$, then $X \equiv 0$.

Lemma 3. *The region* Ω *of existence of the solution of* (2.4) *in the plane* $\mathbb{R}^2(x, X)$ *is bounded by discriminant curves. Exactly four integral curves of Eq.* (2.4) *pass through each point* Ω.

Proof. Because the discriminant curve is given in the plane $\mathbb{R}^2(x, X)$ by the equations $F = 0$ and $F_p = 0$, for Eq. (2.4) we have a criminant which consists of two components in the jet space $\mathbb{R}^3(x, X, p)$:

$$K_1:\quad p = 0,\quad X = 0;\qquad K_2:\quad \alpha_0^4 x^2 X^2 = 1,\quad X \neq 0,\quad p^2 = \alpha_0^2 x X^2. \tag{3.1}$$

The components (3.1) correspond to the discriminant curves

$$DK_1:\quad X = 0;\qquad DK_2:\quad \alpha_0^4 x^2 X^2 = 1,\quad X \neq 0. \tag{3.2}$$

Equation (2.4) is biquadratic in p and can be resolved in the form

$$p = \varepsilon_1 \sqrt{\alpha_0^2 x X^2 + \varepsilon_0 X \sqrt{\alpha_0^4 x^2 X^2 - 1}} \tag{3.3}$$

($\varepsilon_0, \varepsilon_1, \varepsilon_2 = \pm 1$). Real solutions of Eq. (2.4) exist only in the region Ω: $\alpha_0^4 x^2 X^2 \geq 1$, and at each point Ω, Eq. (2.4) has exactly four solutions for p. This implies that exactly four integral curves of Eq. (2.4): C_{++}, C_{+-}, C_{-+}, and C_{--} pass through each point of the region Ω. Lemma 3 is proved.

Below, the manifolds K_1 and DK_1 are not considered since they have no physical meaning [on these manifolds, $\rho = 0$, according to (1.10)]. Since the discriminant curve is the boundary of the region of existence of the solution of (2.4), we denote DK_2 as $\partial\Omega$ (Fig. 2).

The following theorem, proved in [7], is valid.

Theorem 2. *For an even* ν, *the regular singular point* $T_0 = (x_0, y_0, p_0)$ *of Eq.* (2.4) *is a stop point if* $(\partial^\nu F/\partial p^\nu)G > 0$ *and it is a branch point if* $(\partial^\nu F/\partial p^\nu)G < 0$, *where* ν *is such that* $(\partial F/\partial p) = 0, \ldots,$ $(\partial^{\nu-1} F/\partial p^{\nu-1}) = 0$, $(\partial^\nu F/\partial p^\nu) \neq 0$, *and* $G = F_x + pF_y$. *If* ν *is odd, then* T_0 *is a uniqueness point.*

Lemma 4. *The discriminant curve of the key equation contains both regular and irregular singular points.*

1. *Each point of the discriminant curve that is not an irregular singular point is a branch point or a stop point for the integral curves of Eq.* (2.4).

2. *Equation* (2.4) *has two irregular singular points of the type of a focus, whose position is defined by the parameter* α_0.

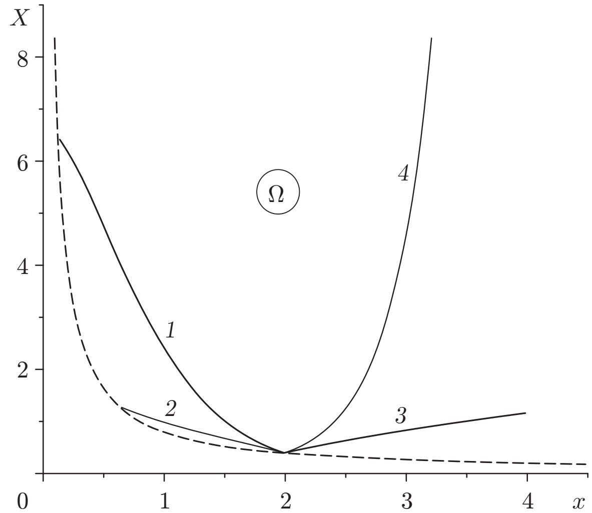

Fig. 2. Discriminant curve $\partial\Omega$ for $X > 0$ (dashed curve) and integral curves of Eq. (2.4) (solid curves) C_{-+} (1), C_{--} (2), C_{+-} (3), and C_{++} (4).

Proof. According to (3.1), $F_{pp} = 8p^2 \neq 0$ on the manifold K_2. By Theorem 1, $\nu = 2$ for Eq. (2.4); in this case, $G \equiv F_x + pF_y = p(2X - 4\alpha_0^2 xXp^2) - 2\alpha_0^2 X^2 p^2$. Then, the condition $G > 0$ specifies the stop points on the criminant. The set of solutions of this inequality is given by

$$\{-1/\alpha_0^{4/3} < X < 0,\ p > 0\} \cup \{0 < X < 1/\alpha_0^{4/3},\ p < 0\}. \tag{3.4}$$

Similarly, the condition $G < 0$ specifies the branch points on the criminant:

$$\{X < -1/\alpha_0^{4/3},\ p > 0\} \cup \{X > 1/\alpha_0^{4/3},\ p < 0\}. \tag{3.5}$$

Inequalities (3.4) and (3.5) define the regular points of the criminant that are branch or stop points. The first item of Lemma 4 is proved.

If $G = 0$, the system of three equations $F = 0$, $F_p = 0$, and $G = 0$ defines a discrete set of points on the criminant and has two solutions in the jet space:

$$\xi_1\colon (1/\alpha_0^{2/3}, -1/\alpha_0^{4/3}, 1/\alpha_0^{2/3}), \qquad \xi_2\colon (1/\alpha_0^{2/3}, 1/\alpha_0^{4/3}, -1/\alpha_0^{2/3}).$$

The change of coordinates

$$x^1 = x - x_0, \qquad X^1 = X - X_0 - X_0'(x - x_0), \qquad p^1 = p - p_0$$

transforms the irregular singular point with the coordinates (x_0, y_0, p_0) into the coordinate origin $O(0,0,0)$. In the neighborhood of the point O, Eq. (2.4) reduces to the differential equation of the first approximation [8]

$$\alpha(X^1)^2 + \beta(x^1)^2 + \gamma x^1 p^1 + (p^1)^2 = 0, \tag{3.6}$$

where

$$\alpha = 2F_{X^1}/F_{p^1p^1}, \quad \beta = F_{x^1x^1}/F_{p^1p^1}, \quad \gamma = 2F_{x^1p^1}/F_{p^1p^1}.$$

We introduce the following notation: $\Delta = -4\beta + \gamma(\alpha + \gamma)$ and $\delta = (\alpha + 2\gamma)^2 - 16\beta$. In [8], the following theorem is proved.

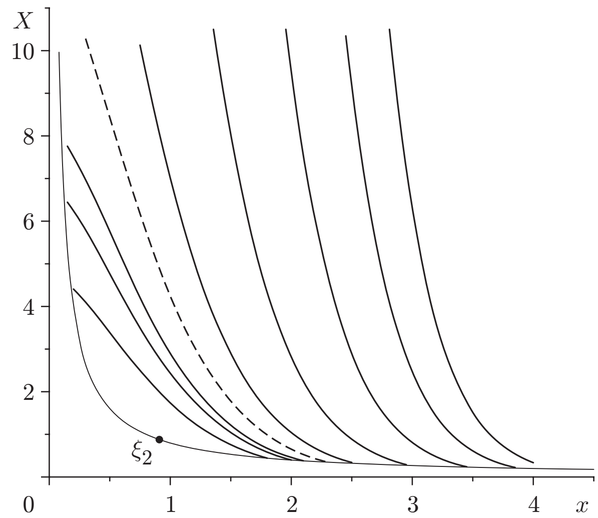

Fig. 3. Integral curves of Eq. (2.4) (solid curves), the separatrix (dashed curve), and the irregular singular point ξ_2 of the key equation on the discriminant curve.

Theorem 3. *For an irregular singular point O of Eq.* (3.6)*, the following classification is valid:*

1) *If $\delta > 0$ and $\Delta < 0$ or if $\delta > 0$ and $\alpha + 2\gamma \neq 0$, the point O is a node;*
2) *If $\delta > 0$ and $\Delta > 0$, the point O is a saddle;*
3) *If $\delta < 0$ and $\alpha + 2\gamma \neq 0$, the point O is a focus;*
4) *If $\delta < 0$ and $\alpha + 2\gamma = 0$, the point O is a center.*

Since $F_{xp} = -4\alpha_0^2 p(X^2 + 2pxX)$, $F_{xx} = 2p^2(1 - \alpha_0^2(4pX + 2p^2x))$, and $F_X = 2X(1 - 2\alpha_0^2 xp^2)$, for the points ξ_1 and ξ_2, we obtain $\alpha + 2\gamma = \pm 5/2$, $\delta = -23/4$, and $\Delta = -3/2$. According to Theorem 2, the points ξ_1 and ξ_2 (Fig. 3) are irregular singular points of the type of a focus. Property 3 of Theorem 3 is proved.

The question of the existence of integral curves that start from the discriminant curve and do not return to it is a fundamental one. Figure 3 gives integral curves of three types: 1) integral curves lying to the left of the dashed curve and defined only on a finite interval of X; 2) curves located to the right of the dashed curve and continuing infinitely on X; 3) the dashed curve (further called the separatrix), which separates the regions in which the integral curves of the two types described above are defined. This curve is the integral curve which corresponds to a certain limiting flow regime. In Sec. 5, it is proved that all these regimes indeed occur.

We note that the irregular singular points of the key equation disappear (move to infinity) as $g_0 \to 0$, which agrees with the problem studied in [3], in which irregular singular points are absent if a constant force is absent.

4. Behavior of the Integral Curves of the Key Equation at Infinity. For a more complete description of the integral curves, it is necessary to elucidate whether integral curves defined for any values of x exist and, if they do exist, to investigate their asymptotic behavior as $x \to \infty$. In (2.4), we transform from the variables (x, X, X') to the new variables (t, q, q'), where $t^2 = 1/x$ and $q^2 = 1/(x^2y^2)$. In the new variables, Eq. (3.3) becomes

$$t^4 q' = 2t^3 q + 2\sqrt{2}\,\alpha_0\varepsilon_1\varepsilon_2 q(1 - q^2/(8\alpha_0^4)) + O(t^2, q^2) \quad \text{at} \quad \varepsilon_0 = 1; \tag{4.1}$$

$$t^4 q' = t^3 q + \sqrt{2}\,\varepsilon_1 q^2/\alpha_0 + o(t^2, q^2) \quad \text{at} \quad \varepsilon_0 = -1. \tag{4.2}$$

For the further analysis of the solution, only the written explicitly leading terms on the right sides of Eqs. (4.1) and (4.2) are important. This change transforms the discriminant curve into the straight lines $q^2 = \alpha_0^4$, which considerably simplifies the analysis. Equation (4.2) can be written as

$$t^{h_0} q' = a_0 q^{m_0}(1 + \varphi_0(q)) + f_0(t, q), \tag{4.3}$$

where $h_0 = 4$, $m_0 = 1$, $a_0 = 2\sqrt{2}\,\varepsilon_1\beta_0$, and φ_0 and f_0 are holomorphic functions of their arguments ($\beta_0 = \text{const} > 0$). According to the Poincaré–Bendixson theory [9, 10], the neighborhood of the point $(0, 0)$ for Eq. (4.3) for specified values of the parameters h_0, m_0, and a_0 is divided into three sectors: two hyperbolic and one parabolic. The behavior of the solution for $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1$ is presented in Fig. 4.

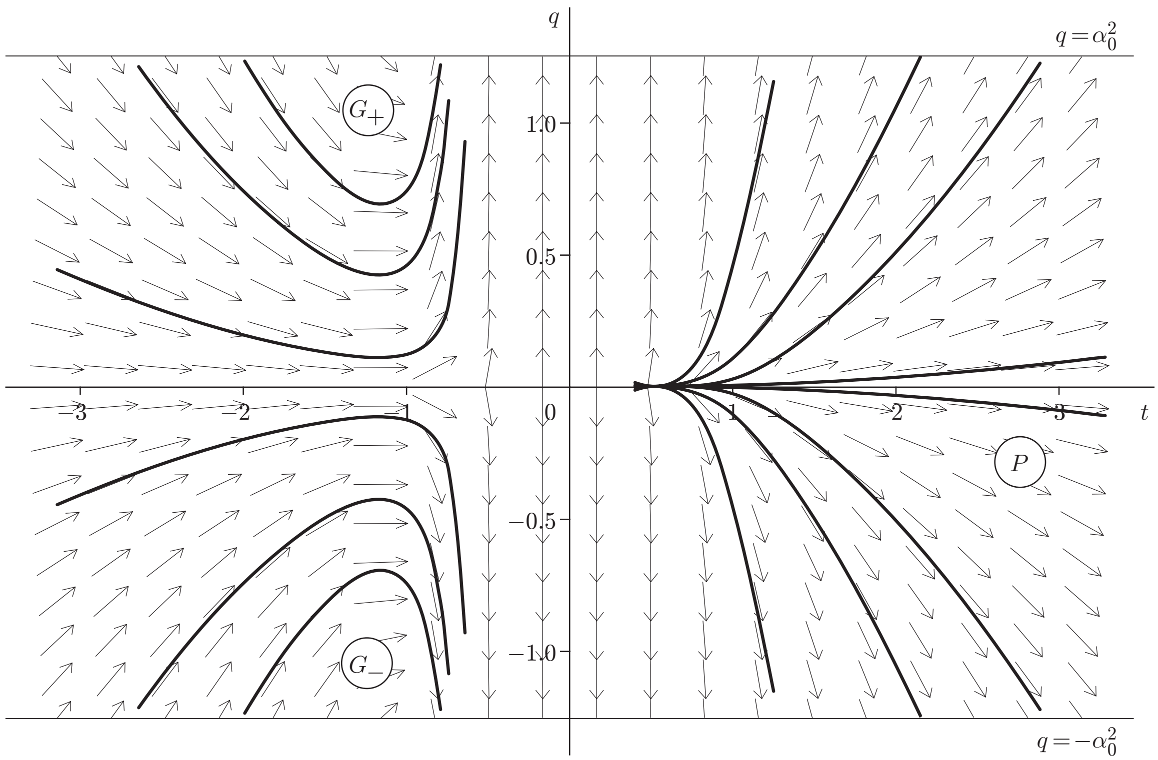

Fig. 4. Behavior of integral curves of (4.3) in the neighborhood of the coordinate origin.

Thus, for certain initial data, some integral curves of Eq. (4.3) enter the point $O(0,0)$, and, consequently, the integral curves of the basic equation (2.4) exist for any values of $X > 0$ (the parabolic sector P). The integral curves lying in the hyperbolic sectors G_+ and G_- do not reach the point $O(0,0)$, and, hence, their corresponding integral curves of Eq. (2.4) exist on a finite interval of X. The separatrix corresponds to the straight-line segment $\{t = 0,\ q^2 < \alpha_0^4\ \}$. This behavior of the integral curves in the neighborhood of the singular point allows a conclusion to be drawn on the asymptotic behavior of the gas motion. Depending on the start conditions, gas particle motion can persist over infinitely long distances in the direction x or it can occur only on a finite interval of x an. This depends on the ratio of the kinetic and potential energies.

For $\varepsilon_0 = -1$ in Eq. (4.3), we have $h_0 = 4$, $m_0 = 2$, and $a_0 = \sqrt{2}\,\varepsilon_1/\alpha_0$. In this case, the neighborhood of the singular point is divided into two hyperbolic and one or two parabolic sectors. The data of the numerical experiment confirm the existence of two hyperbolic and two parabolic sectors.

5. Characteristics and Sonic Line. We investigate the question of in which regions the flow described by formulas (1.10) with $H \equiv 0$ is subsonic and supersonic.

Lemma 5. *The points of the discriminant curve* (3.2) *in the plane* $\mathbb{R}^2(x, X)$ *are images of the invariant sonic characteristics of system* (1.10) *defined in physical space by the equation* $x = x_0$. *On the discriminant curve,* $u^2 = c^2$.

Proof. Writing the condition $u^2 = c^2$ in terms of the function σ: $1/\sigma' = c_0^2(\sigma'/\sigma)^2$, we obtain the equation $p^2 = X$, which for the solutions of the key equation (2.4) has the form $\alpha_0^2 x X = 1$. Lemma 5 is proved.

For $H \equiv 0$, solution (1.10) defines the two-dimensional steady-state gas flow in the plane $\mathbb{R}^2(x, y)$. In this case, the following lemma holds.

Lemma 6. *The integral curves of Eq.* (2.4) *coincide with the streamlines of the gas flow in the physical plane.*

Proof. The streamlines in the physical plane are defined by the equality $dx/u(x, y) = dy/v(x, y)$, which in terms of the function σ has the form $\sigma' dx = \sigma dy/y$. Integration yields

$$y = y_0\sigma(x), \qquad y_0 = \text{const}. \tag{5.1}$$

The solution of the key equation (2.4) has the form $X = X(x)$. In view of change (2.4), equality (5.1) implies that in the physical plane $\mathbb{R}^2(x, y)$, x and y are related in the same manner as x and X in the phase plane $\mathbb{R}^2(x, X)$. Lemma 6 is proved.

Lemma 7. *The integral curves C_{+-} and C_{--} for $\varepsilon_0 = -1$ correspond to supersonic gas flows, and for them the following inequality holds:*

$$u^2 > c^2. \tag{5.2}$$

The integral curves C_{++} and C_{-+} for $\varepsilon_0 = 1$ correspond to gas flows for which transition through the sound velocity is possible, and for them the following inequality holds:

$$u^2 < c^2. \tag{5.3}$$

Proof. In view of (2.4), expression (5.2) is equivalent to the inequalities

$$\{-X < p^2 < X,\ X > 0\} \cup \{X < p^2 < -X,\ X < 0\}. \tag{5.4}$$

Substituting p^2 from (3.3) into (5.4), in the region Ω of existence of the solution of inequality (5.4), we obtain

$$\Omega: \quad \alpha_0^2 x X > 1. \tag{5.5}$$

Inequalities (5.4) have no solutions for the integral curves C_{++} and C_{-+} and are satisfied identically for the integral curves C_{+-} and C_{--}. Similarly, we find that the system of inequalities (5.3) and (5.5) is satisfied identically for the integral curves C_{++} and C_{-+} and is not compatible for the integral curves C_{+-} and C_{--}. This implies that the integral curves C_{+-} and C_{--} describe supersonic flow. For the integral curves C_{++} and C_{-+}, inequality (5.3) is satisfied, and, therefore, on these curves, transition through the sound velocity is possible.

Lemma 8. *For the gas flow defined by solution* (1.10) *for $H \equiv 0$, the image of the sonic line S: $|\boldsymbol{u}| = c$ in the plane $\mathbb{R}^2(x, y)$ is given by the equation*

$$y^2 = (\sigma(x)^2 A_0^2 - c_0^2)/A_0,$$

where x is the initial physical variable, $\sigma(x)$ is a solution of the key equation (2.4) *in the variables $(x, \sigma(x))$, and $A_0 = g_0 x + b_0$ is the total energy of the particle.*

Proof. Substituting the equation of the sonic line $u^2 + v^2 = c^2$ into the Bernoulli integral $u^2 + v^2 + c^2 = 2(g_0 x + b_0)$, we obtain the system of two equations

$$u^2 + v^2 = g_0 x + b_0, \qquad c^2 = g_0 x + b_0. \tag{5.6}$$

The second of them has the form $(\sigma')^2 = \sigma^2(g_0 x + b_0)/c_0^2$. Substitution of this expression into the first equation of (5.6) written in terms of (1.10) yields $(1/\sigma')^2 + y^2/\sigma^2 = g_0 x + b_0$. As a result, we have

$$y^2 = (\sigma(x)^2 (g_0 x + b_0)^2 - c_0^2)/(g_0 x + b_0),$$

quod erat demonstrandum.

The numerical experiment shows that the image of the sonic line is in the field of existence of the solution in the plane $\mathbb{R}^2(x, X)$.

6. Flows with a Stationary Shock Wave. We consider gas flow with a fixed shock wave, whose front is defined by the equation $x = x_0$. Let p_i, ρ_i, and c_i be the pressure, density, and sound velocity, respectively, and u_i the gas velocity components normal to the front ahead of the shock front ($i = 1$) and behind it ($i = 2$). Then, for a polytropic gas with the equation of state $p_i = S_{0i}\rho^3$, the following conditions at the discontinuity — the Rankine–Hugoniot relations — hold [2]:

$$\rho_1 u_1 = \rho_2 u_2; \tag{6.1}$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_1^2; \tag{6.2}$$

$$3p_1/\rho_1 + u_1^2 + g_0 x_0 = 3p_2/\rho_2 + u_2^2 + g_0 x_0. \tag{6.3}$$

The velocity components tangential to the front are conserved in passing through the discontinuity:

$$v_1 = v_2. \tag{6.4}$$

The Zemplén theorem holds: the absolute value of the velocity component u_i normal to the shock front is higher than the sound velocity c_i ahead of the shock front and lower than the sound velocity behind the front:

$$u_1^2 > c_1^2, \qquad u_2^2 < c_2^2.$$

The statement of the Zemplén theorem is equivalent to the statement that the entropy increases in passing through the discontinuity:

$$S_{02} > S_{01}.$$

We write formulas (1.10) for the states on the opposite sides of the discontinuity in terms of the function $X = X(x)$ for $H \equiv 0$:

$$u_i = \frac{c_{0i}}{X_i'}, \quad v_i = \frac{yc_{0i}}{X_i}, \quad w = W_0, \quad \rho_i = R_{0i}\,\frac{X_i'}{X_i}, \quad S_i = S_{0i}, \quad p_i = S_{0i}\rho_i^3. \tag{6.5}$$

In this case, condition (6.4) becomes

$$X_2 = (c_{02}/c_{01})X_1. \tag{6.6}$$

Since $\rho u = R_0 c_0/X$, from (6.6) and (6.1) it follows that R_{0i} is conserved:

$$R_{01} = R_{02} = R_0. \tag{6.7}$$

Substitution of $c_0^2 = 3S_0R_0^2$ into (6.6) yields

$$X_2 = \sqrt{S_{02}/S_{01}}\,X_1. \tag{6.8}$$

We substitute the solution representation (6.5) into the expression $p + \rho u^2$ and replace the derivatives of the function $X(x)$ from (4.3). By virtue of (6.7) and (6.8), the obtained equality is simplified and becomes

$$\frac{D_1^4 + X_1^2}{D_1X_1} - \frac{D_2^4 + X_2^2}{D_2X_2} = 0, \tag{6.9}$$

where

$$D_i = \sqrt{\frac{g_0x_0}{3S_{01}R_0^2}\,X_1^2 + (-1)^i\sqrt{\frac{g_0^2x_0^2}{9S_{01}^2R_0^4}\,X_1^4 - X_i^2}}.$$

The condition of energy conservation at the discontinuity (6.3) coincides with the Bernoulli invariant integral (1.9) and, hence, with Eq. (2.4) and does not impose additional constraints: the conservation of the constant b_0 follows from the general Bernoulli integral, and the term g_0x is constant at the discontinuity. Thus, the key equation (2.4) is condition (6.3) at the discontinuity. The arbitrariness in the choice of the solution ahead of and behind the shock front is determined by the arbitrariness in the choice of the integral curves of Eq. (4.3).

We summarize the obtained results in the form of the following statement.

Theorem 4. *For the solution of* (6.5) *and* (2.4)*, conditions* (6.1)–(6.3) *at the discontinuity are equivalent to the finite relations* (6.8) *and* (6.9)*, which link the values of the solutions* X_1 *and* X_2 *of the differential equation* (2.4) *at the discontinuity front* $x = x_0$.

This result is also valid for motion in the absence of the force [11].

Relation (6.8) specifies a one-parameter family of straight lines with the slope $\mathbb{R}^2(X_1, X_2)$ in the plane of states $s = \sqrt{S_{01}/S_{02}}$. According to the Zemplén theorem, $0 < s < 1$. Equation (6.9) defines a certain curve in this plane. Each straight line (6.8) corresponds to the class of shock transitions, the self-conjugation of solutions of the form (6.5) with the specified ratio $S_{01}/S_{02} = s^2$. The points of intersection of straight line (6.8) with curve (6.9) define the pairs of states (X_1, X_2) conjugate through the shock wave. Thus, curve (6.9) can be called the shock adiabat that characterizes the possible shock transitions for the given solution (6.5).

7. Analysis of the Shock Adiabat. Because the key equation (2.4) admits the involution $J\colon X \mapsto -X$, and the regions of definition of the solution for $X > 0$ and $X < 0$ do not have common points, we assume, without loss of generality, that $X_1 > 0$ and $X_2 > 0$.

Transformation of equality (6.9) yields

$$kX_1(1 - X_1^2/X_2^2) = \sqrt{kX_1^2 - X_2^2/X_1^2} + \sqrt{kX_1^2 - 1}, \tag{7.1}$$

where $k = g_0 x_0/(3S_{01}R_0^2)$. In the above form formula (7.1) is inconvenient for investigation. We make the change of variables $(X_1, X_2) \to (t, q)$, where

$$t = X_2/X_1, \qquad q = X_1. \tag{7.2}$$

In the variables (7.2), Eqs. (7.1) is written as

$$kq(1 - 1/t^2) = \sqrt{kq^2 - t^2} + \sqrt{kq^2 - 1}. \tag{7.3}$$

Equation (7.3) is resolved uniquely for the variable q:

$$q = \frac{t^2}{k}\sqrt{-\frac{k^2t^2 + k^2 + 2\sqrt{k^4t^2 + k^3t^4}}{k(-4t^4 + kt^4 - 2kt^2 + k)}}. \tag{7.4}$$

This is valid because $q > 0$, and, hence, of the four possible roots of Eq. (7.3), two negative roots may not be considered. By virtue of the Zemplén theorem and (6.8), we obtain $t > 1$. This implies that the region of the possible shock transitions is $\Delta = \{(t, q)\colon\ t > 1,\ q > 0\}$. In the region Δ, of the two remaining roots, the real root is the one defined by formula (7.4). We note that as the value of k changes, the function $q(t)$ defined by formula (7.4) changes behavior. Indeed, the values of

$$t_+ = 1\Big/\sqrt{1 - 2/\sqrt{k}}, \qquad t_- = 1\Big/\sqrt{1 + 2/\sqrt{k}} \tag{7.5}$$

are roots of the denominator of the radicand and depend significantly on the value of the parameter k; the root $t_- < 1$ for any values of k. However, $t_+ \neq 0$ $(t_+ \in i\mathbb{R})$ for $k < 4$ and $t_+ > 1$ for $k > 4$; for $k = 4$, the denominator, which is generally a fourth-order polynomial in t, becomes a square binomial. With the change (7.2), the straight lines (6.8) are given by the equation

$$t = \text{const}. \tag{7.6}$$

7.1. *Case* $k < 4$. According to (7.5), for $k < 4$, solution (7.6) exists for all $t > 1$. In addition, the following lemma is valid.

Lemma 9. *In the case* $k < 4$*, solution* (7.4) *has the following asymptotic behavior as* $t \to \infty$:

$$q \sim \frac{t}{k}\sqrt{\frac{k + \sqrt{k}}{4 - k}}.$$

Proof. In (7.4), it is only necessary to insert t^2 under the radical and let t to infinity.

7.2. *Case* $k > 4$. For $k > 4$, the denominator in (7.4) has a second positive root, which, in addition, is larger than unity. Since the numerator of the radicand is strictly larger than zero for all values of t, real solutions (7.4) exist only for $t \in (t_-, t_+)$. In this case, the following lemma, which is an analog of Lemma 9, is valid.

Lemma 10. *In the case* $k > 4$*, solution* (7.4) *has the following asymptotic behavior as* $t \to t_+$:

$$q \sim F_0/\sqrt{t_+ - t^2}, \qquad F_0 = \text{const}.$$

The proof is similar to the proof of Lemma 9.

7.3. *Case* $k = 4$. In the case $k = 4$, we write (7.4) as

$$q = \frac{t^2}{4}\sqrt{\frac{t^2 + 1 + \sqrt{t^4 + 4t^2}}{2t^2 - 1}}. \tag{7.7}$$

The following statement is valid.

Lemma 11. *The function* $q(t)$ *defined by* (7.7) *has the following asymptotic behavior as* $t \to \infty$:

$$q \sim t^2/4.$$

The proof is similar to the proof of Lemma 9.

The behavior of the shock adiabat and the straight line (7.6) differs from the case $k < 4$ only in that the adiabat has a different asymptotic behavior as $t \to \infty$.

Using the results obtained above, we formulate the following theorem.

Theorem 5. *Any state ahead of the discontinuity* (X_1, S_{01}) *corresponds to a pair* (X_2, S_{02})*, where* X_2 *is calculated from Eq.* (6.9) *and* S_{02} *is calculated according to* (6.8).

Proof. The three cases considered above describe all possible versions of the behavior of the shock adiabat. In each of them, it has a single (single-valued) branch of type (7.4), which, by virtue of Lemmas 9–11, has one and only one point of intersection with the straight line (7.6) on the plane (t, q). This implies that the shock adiabat assigns exactly one value of $t^* = X_2/X_1 > 1$ to each value of $q^* = X_1$. Thus, the state X_2 behind the discontinuity is calculated by the formula

$$X_2 = q^*/t^*.$$

Then, we substituted the pair (X_1, S_{01}) into (6.9) and solve the obtained equality for X_2. Then, substituting (X_1, S_{01}) and X_2 into (6.8), we uniquely calculate S_{02}. Theorem 5 is proved.

8. Flow Pattern in the Physical Plane. Let us give a physical interpretation of the solution obtained.

8.1. *Interpretation of the Continuous Solution.* In formula (3.3), the parameter ε_1 specifies the flow regimes (a value of $+1$ corresponds to the source and a value of -1 to the sink). The flow pattern is determined by how the gravity and the motion of the gas jet are oriented relative to each other. If they are unidirectional, the source occurs; if they are opposite in direction, the sink takes place.

In case of the source, the following regimes of motion are possible.

1. The gas jet is accelerated and moves to infinity.
2. The gas jet is accelerated and moves to infinity with a different asymptotic behavior (the separatrix case).

These regimes correspond to integral curves of the type C_{+-}, and the second case is the limiting case of the first one (the dashed curve in Fig. 3).

3. The particles stop at a finite distance from the point of issue (flow of particles from one surface onto the other). The stop occurs in spite of the fact that the jet moves in the direction of the gravity force. The reason of this is that the motion of gas particles differs from the motion of material particles by the presence of the resistance of the medium — pressure. Gas particles expend additional energy to move forward. The start point of such particles cannot be arbitrary: it is related to the value of the problem parameter α_0 (see Lemma 3).

In the case of the sink, the gas motion occurs as follows.

1. The gravity force stops the gas particle flow coming from infinity.
2. The gravity force stops the gas particle flow coming from infinity with a different asymptotic behavior (the separatrix case).

Both regimes correspond to integral curves of the type C_{-+}. Again, the second case is the limiting case of the first one. From a physical point of view, there is a certain condensation surface, on which the particles that arrive are accumulated and which plays the role of a surface of zero potential level.

3. Stop of the gas particles at a finite distance from the sink is possible.

8.2. *Interpretation of the Strong-Discontinuity Solution.* According to formula (5.1), the behavior of the integral curves of the key equation (2.4) is similar to the behavior of the streamlines of steady-state plane gas flow (1.10). Figure 5 shows a shock transition in the physical plane at $x = x_0$ (curve 2 is an integral curve C_{+-} that defines the flow ahead of the shock wave, curve 3 is an integral curve C_{++} that defines the flow behind the shock wave). The point of intersection of these integral curves in the plane $\mathbb{R}^2(x, X)$ corresponds to the shock wave. The motion of the gas particles in space corresponds to the motion of a point in the plane (x, X) along the integral curves. At $x < x_0$, the point moves along curve 2, and at $x > x_0$, it goes over into curve 3.

Conclusions. The study of the submodel showed that the region of existence of the solution in the phase plane $\mathbb{R}^2(x, X)$ is bounded by a discriminant curve. All regular points of the key differential equation were considered and classified. The existence of irregular singular points was established and shown to significantly complicate the investigation of the model and qualitatively change the flow pattern.

From a physical point of view, the model describes two-dimensional steady-state gas flow in a constant force field. In this case, the discriminant curve is a source for $p > 0$ and a sink for $p < 0$. The asymptotic behavior of the solution as $x \to \infty$ was studied.

The sonic characteristics of system (1.10) were studied, and the equation of the sonic line was obtained. It was proved that the sonic line is located in the region Ω of existence of the solution. During passage of the gas flow along the integral curves C_{++} and C_{+-} in this region, a continuous transition from supersonic to subsonic flow occurs. Analogs of the shock-adiabat equation and the Michelson straight line were obtained. A feature of the problem in question is the strong dependence of the behavior of the shock adiabat on the problem parameter g_0, the position of the shock front x_0, and the value of the entropy ahead of the front S_{01}.

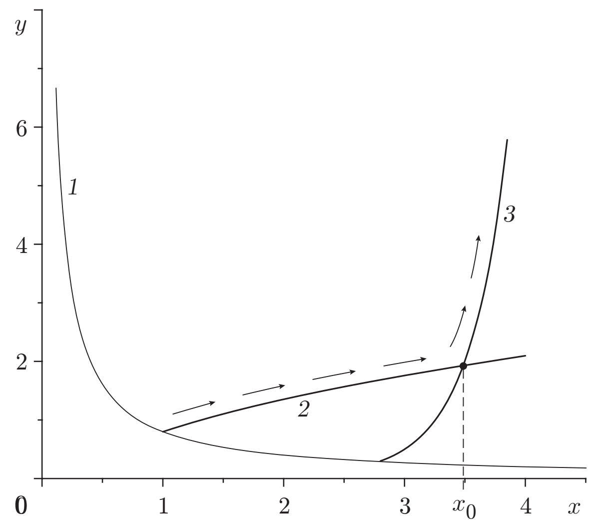

Fig. 5. Curves in the plane (x, y) that define a shock transition: the boundary of the region of existence of the solution (1) and integral curves C_{+-} (2) and C_{++} (3); the point corresponds to the shock transition.

Because the arguments H are Lagrangian coordinates and are continuous in passing through the discontinuity $x = x_0$, the given construction of the shock transition is extended to general solutions of the form (1.1), in which the function $H \neq 0$.

Of interest are the physical interpretation of the irregular singular points ξ_1 and ξ_2 and the construction of a shock-wave solution for the maximally general gas equation of state. In this case, the nonlinear dependence on the derivative in the key equation (2.4) is defined by the equation of state [the function $I = I(\rho)$].

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